Dynamics and the clopen type semigroup: some remarks and many questions

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Motivation: a theorem of Giordano–Putnam–Skau

Theorem (Giordano–Putnam–Skau 1995)
Let φ, ψ be two minimal homeomorphisms of the Cantor space X.
Then the following conditions are equivalent:

- φ and ψ are *orbit equivalent*, i.e. $\exists g \in \mathrm{Homeo}(X)$ s.t.

$$\forall x, y \in X \ (\exists n \in \mathbb{Z} \ \varphi^n(x) = y) \Leftrightarrow (\exists i \in \mathbb{Z} \ \psi^i(g(x)) = g(y))$$

- There exists $g \in \mathrm{Homeo}(X)$ such that

$$\{g_*\mu \colon \mu \text{ is } \varphi\text{-invariant}\} = \{\nu \colon \nu \text{ is } \psi\text{-invariant}\}$$

(above μ, ν range over all Borel probability measures on X)

One direction is almost immediate; the other motivates us to investigate sets of invariant measures (first for $\mathbb{Z}$-actions, then for other groups).

(Very) open problems

Let G be countable amenable.

- Is every minimal Cantor G-action orbit equivalent to a $\mathbb{Z}$-action?
- Given a minimal Cantor G-action α, does there exist a minimal Cantor $\mathbb{Z}$-action which preserves the same Borel probability measures as α?

Sets of invariant measures for minimal Cantor $\mathbb{Z}$-actions

Theorem (Ibarlucía–M. 2015)
Let X be a Cantor space and $K \subset \mathbb{P}(X)$. There exists a minimal homeomorphism φ of X s.t. $K = \{\mu : \mu \text{ is } \varphi\text{-invariant}\}$ iff:

- K is compact and nonempty.
- Every element of K is atomless and has full support.
- $\forall A, B \in \mathrm{Clopen}(X)$ s.t. $\mu(A) < \mu(B)$ for all $\mu \in K$, $\exists C \in \mathrm{Clopen}(X)$ s.t. $C \subset B$ and $\mu(A) = \mu(C)$ for all $\mu \in K$.

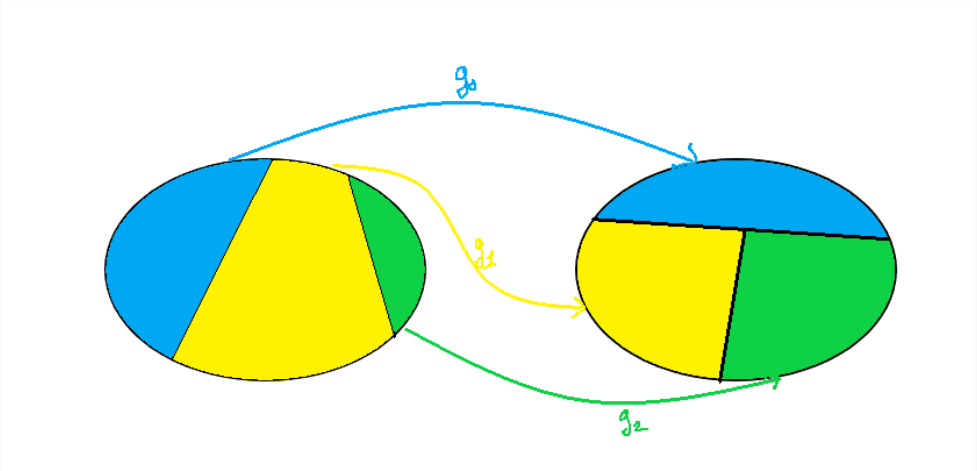
The first two conditions are obviously necessary for any minimal action of any amenable group.

Necessity of the last one comes from a result of Glasner–Weiss using properties of the *topological full group*.

The topological full group

From now on G is an infinite countable group; X is compact, 0-dimensional, Hausdorff.

Definition
Given an action $\alpha\colon G \curvearrowright X$, the *topological full group* $[[\alpha]]$ is the subgroup of all homeomorphisms γ for which there exists a clopen partition $(U_i)_{i\in I}$ s.t. γ coincides with some $g_i \in G$ on each U_i.



Dynamical comparison

Definition (Buck 2013, Kerr 2017)
$\alpha\colon G \curvearrowright X$ has *dynamical comparison* if :

$$\forall A, B \in \mathrm{Clopen}(X)^* \, (\forall \mu \in \mathbb{P}(\alpha) \, \mu(A) < \mu(B)) \Rightarrow \exists \gamma \in [[\alpha]] \, \gamma A \subset B$$

If a minimal Cantor action of an amenable group has dynamical comparison, then there exists a minimal $\mathbb{Z}$-action with the same invariant Borel probability measures.

Originally proved for minimal $\mathbb{Z}$-actions by Glasner and Weiss (1995), this property is now known to hold for

- Actions of groups of local subexponential growth (Downarowicz–Zhang 2019).
- Free actions of elementary amenable groups (Kerr-Narishkyn 2021).

The clopen type semigroup

Say that $A \subseteq X \times \mathbb{N}$ is *bounded* if $A \cap (X \times \{n\}) = \emptyset$ for large enough n.

Fix $\alpha\colon G \curvearrowright X$. For a bounded, clopen $A \subset X \times \mathbb{N}$ let $[A]$ denote the (clopen) equidecomposability class of A for the natural action $\tilde{\alpha}\colon G \times \mathfrak{S}(\mathbb{N}) \curvearrowright X \times \mathbb{N}$.

Then set $[A] + [B] = [\tilde{A} \sqcup \tilde{B}]$ where $[\tilde{A}] = [A]$, $[\tilde{B}] = [B]$ and $\tilde{A} \cap \tilde{B} = \emptyset$.

We obtain a commutative monoid which I denote $T(\alpha)$; it is a *refinement monoid*: if $x_1 + x_2 = y_1 + y_2$ then there exist $z_{i,j}$ such that $x_i = z_{i,1} + z_{i,2}$ and $y_j = z_{1,j} + z_{2,j}$ for $i, j \in \{1, 2\}$.

We order $T(\alpha)$ using the algebraic preorder:

$$(a \leq b) \Leftrightarrow (\exists c\ a + c = b)$$

States and invariant measures

Definition
A *state* is a morphism $\mu\colon (T(\alpha), +) \to [0, +\infty]$. It is *normalized* if $\mu([X]) = 1$. I denote by $\mathbb{P}(\alpha)$ the set of normalized states.

Normalized states correspond to G-invariant Radon probability measures on X, via the relation $\tilde{\mu}(A) = \mu([A])$; I will not make the distinction below.

Theorem (particular case of Tarski 1929)
Given $a \in T(\alpha)$, there exists a state μ such that $\mu(a) = 1$ iff $(n+1)a \not\leq na$ for all $n \in \mathbb{N}$.

Dynamical comparison seen in $T(\alpha)$

Proposition
$\alpha\colon G \curvearrowright X$ has dynamical comparison iff

$$\forall a, b \in T(\alpha)^* \, (\forall \mu \in \mathbb{P}(\alpha) \; \mu(a) < \mu(b)) \Rightarrow (a \leq b)$$

A closely related statement appears in Ara–Bönicke–Bosa–Li (2020); the proof is already present in Downarowicz–Zhang (2019).

Definition
$T(\alpha)$ is *almost unperforated* if

$$\forall a, b \in T(\alpha) \; \forall n \in \mathbb{N} \quad (n+1)a \leq nb \Rightarrow a \leq b$$

Dynamical comparison and almost unperforation

Proposition

- Assume that G is amenable. Then α has dynamical comparison iff for every order unit $b \in T(\alpha)$ and every a such that $(n+1)a \leq nb$ for some n, one has $a \leq b$.
- If α is minimal then α has dynamical comparison iff $T(\alpha)$ is almost unperforated (and if $\mathbb{P}(\alpha) = \emptyset$ then $a \leq b$ for all nonzero $a, b \in T(\alpha)$).

This statement has a precursor in Kerr (2018), a version valid for all compact metrizable spaces is given by Ma (2019) and there is a related statement for second-countable ample groupoids by Ara–Bönicke–Bosa–Li (2020). Kerr seems the first to notice the connection with clopen type semigroups.

Weak comparability and cancellativity

On this slide we assume that α is minimal.

Proposition

- α has dynamical comparison iff $T(\alpha)$ has the *weak comparability property*:

$$\forall a \neq 0 \ \exists k \in \mathbb{N}^* \ \forall b \ (kb \leq [X]) \Rightarrow (b \leq a)$$

- If $\mathbb{P}(\alpha) \neq \emptyset$ and α has dynamical comparison then $T(\alpha)$ is cancellative: whenever $u + v = u + w$ one has $v = w$.

This follows from work of Ara–Pardo (1996) and Ara–Goodearl–Pardo–Tyukavkin (1995) on refinement monoids.

Existence of a dense locally finite subgroup in $[[\alpha]]$

Definition
$T(\alpha)$ is *unperforated* if

$$\forall a, b \in T(\alpha)\ \forall n \in \mathbb{N}^* \quad na \leq nb \Rightarrow a \leq b$$

Theorem (M.–Robert)
Assume that $\alpha\colon G \curvearrowright X$ is a minimal action of G on a Cantor space such that $\mathbb{P}(\alpha) \neq \emptyset$. Then $[[\alpha]]$ has a dense, locally finite subgroup iff $T(\alpha)$ is unperforated.

Proof based on work of Ara–Goodearl (2014) on *tame* refinement monoids.

For minimal $\mathbb{Z}$-actions certain dense locally finite subgroups of $[[\alpha]]$ play a key role in the GPS classification.

Saturation

Definition
Say that α is *saturated* if it has dynamical comparison and

$$\forall A, B \in \mathrm{Clopen}(X)^* \ (\forall \mu \in \mathbb{P}(\alpha) \ \mu(A) = \mu(B)) \Rightarrow [A] = [B]$$

Again, this can be formulated as a condition on $T(\alpha)$; for minimal α saturation amounts to the conjunction of dynamical comparison and:

$$\forall a, b \in T(\alpha) \ (\forall c \ (c < a) \Leftrightarrow (c < b)) \Rightarrow (a = b)$$

Some reasons to care: GPS classification of saturated minimal $\mathbb{Z}$-actions is much simpler than the general case. For minimal, saturated $\mathbb{Z}$-actions, $T(\alpha) = K_0(\alpha)^+$.

The Stone-Cech compactification

Theorem

- $T(G \curvearrowright \beta G)$ is unperforated (König 1926).
- $\leq$ is a partial order on $T(G \curvearrowright \beta G)$ (Banach 1924)

Hence βG has dynamical comparison; if G is amenable and $A, B \subset G$ are s.t. $\mu(A) < \mu(B)$ for any G-invariant f.a.p.m. μ, there exist $A_1, \dots A_n$, $g_1, \dots, g_n$ s.t.

$$\bigsqcup_{i=1}^{n} A_i = A \text{ and } \bigsqcup_{i=1}^{n} g_i A_i \subset B$$

$\beta\mathbb{Z}$ is not saturated: there exist A, B such that $\mu(A) = \mu(B) = \frac{1}{2}$ for every $\mathbb{Z}$-invariant f.a.p.m. μ, yet A and B are not equidecomposable (I learnt this from B. Weiss).

The universal minimal flow

Every topological group admits a universal minimal flow μG (a minimal G-flow that factors onto every other minimal G-flow).

For discrete G, any minimal subflow of βG is isomorphic to μG; the action $G \curvearrowright \mu G$ is free; and there is an equivariant retraction $r\colon \beta G \to \mu G$.

Proposition (M.–Robert)
$T(G \curvearrowright \mu G)$ is isomorphic to a submonoid of $T(G \curvearrowright \beta G)$; so it is unperforated and $\leq$ is a partial order on $T(G \curvearrowright \mu G)$.

A consequence on factors

Theorem (M.–Robert)
Every minimal Cantor action is a factor of a minimal Cantor action α such that:

- α is free;
- $T(\alpha)$ is unperforated (hence α has dynamical comparison)
- $\leq$ is a partial order on $T(\alpha)$

Proof: Use the fact that μG has these properties to build a Cantor action $\beta\colon G \curvearrowright Y$ which also has them and s.t. $\pi\colon \mu G \to X$ factors trough Y.

For amenable G, the fact that any minimal action is a factor of a free minimal action with dynamical comparison also follows from work of Conley–Jackson–Kerr–Marks–Seward–Tucker-Drob (2017)

The Polish space of minimal actions

The space $\mathrm{Act}(G)$ of all actions of G on a Cantor X is a G_δ subset of $\mathrm{Homeo}(X)^G$, hence a Polish space.

Minimal actions form a G_δ subset of $\mathrm{Min}(G) \subset \mathrm{Act}(G)$.

What about generic properties in $\mathrm{Min}(G)$? The conjugation action $\mathrm{Homeo}(X) \curvearrowright \mathrm{Min}(G)$ is topologically transitive (for any G)

Thus any Baire measurable, conjugacy invariant subset of $\mathrm{Min}(G)$ is either meagre or comeagre.

Genericity in $\mathrm{Min}(G)$

Proposition (M.–Robert)
The following properties are G_δ in $\mathrm{Min}(G)$; the first five are dense (density follows from our remark on factors and the fact that if β is a factor of α then β belongs to the closure of the conjugacy class of α).

1. Freeness.
2. Dynamical comparison ($= T(\alpha)$ almost unperforated).
3. Unperforation of $T(\alpha)$.
4. Cancellativity of $T(\alpha)$.
5. $\leq$ being a partial order on $T(\alpha)$.
6. Saturation.
7. Unique ergodicity.

Some questions on $\mathrm{Min}(G)$ and $\mathrm{Act}(G)$

For the questions I focus on the minimal Cantor case.

- When does there exist a generic conjugacy class in $\mathrm{Min}(G)$? True for $G = \mathbb{Z}$ (universal odometer, Hochman 2007).
- Closure of $\mathrm{Min}(G)$ in $\mathrm{Act}(G)$? Known for $\mathbb{Z}$: chain-transitive systems (Bezuglyi–Dooley–Kwiatkowski 2006).
- For amenable G, is unique ergodicity generic in $\mathrm{Min}(G)$?
- Does every countable group admit a free, miimal, uniquely ergodic Cantor action? True for amenable G (Rosenthal 1985) Every group admits a free, minimal Cantor action with an invariant probability measure (Elek 2020).

Some questions on $(T(\alpha), +, \leq)$

Again, focus on minimal Cantor actions.

- Is $(T(\alpha), +)$ always cancellative when $\mathbb{P}(\alpha) \neq \emptyset$? (recall that cancellativity follows from dynamical comparison)
- Example of minimal action such that $\leq$ is not a partial order? (Must have $\mathbb{P}(\alpha) = \emptyset$ if α minimal) And for actions that are not necessarily minimal?
- Example of minimal action such that $\mathbb{P}(\alpha) = \emptyset$ but $T(\alpha) \neq \{0, \infty\}$? (generically $T(\alpha)$ is trivial for nonamenable G)

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