Forcing axioms, inner models and determinacy

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Gödel's Program

There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline, and furnishing such powerful methods for solving given problems (and even solving them, as far as that is possible, in a constructivistic way) that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any well established physical theory.

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Gödel's Program: Remove independence from set theory by passing to extensions of ZFC.
Levy-Solovay: This cannot be achieved by the large cardinal hierarchy.

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Assume $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} +$ "Θ *is a regular cardinal". Then* $\mathbb{P}_{\max} * \mathrm{Add}(\omega_3, 1)$ *forces* $\mathsf{MM}^{++}(c)$.

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Thus, an *honest* practitioner of MM cannot dismiss AD and vice a versa.

Steel's Program: my take

Essentially, stop worrying and start proving that all these distinct set theories are the same. As a consequence, you will develop an immensity beautiful mathematics.

Steel's Program, or how i stoped worrying and started doing mathematics

1. (Axiom (*) for CH) Assume ZF $+$ $AD_{\mathbb{R}}$ $+$ "Θ is a regular cardinal". Let $g \subseteq Coll(\omega_1, \mathbb{R})$ be generic. $V[g] \vDash CH$. What properties does $V[g]$ have?

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2. (Adolf-S.-Trang-Wilson-Zeman) Assume the conclusion of Woodin's Theorem. Then the minimal model of $\mathrm{AD}_{\mathbb{R}}$ $+$ "Θ is a regular cardinal"exists.

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3. Does MM^{++} hold in a countably closed homogenous forcing extension of a model of determinacy?
4. Assume MM^{++}. Is there an inner model with a supercompact cardinal?Is there an inner model with a superstrong cardinal? Is there an inner model with a Woodin cardinal that is a limit of Woodin cardinals.

The plan

In this tutorial, we will

1. construct a model of determinacy and
2. force over it to obtain a model satisfying

$$\mathrm{MM}^{++}(c) + \neg\Box_{\omega_3} + \neg\Box(\omega_3).$$

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1. The determinacy model is a type of Chang model. It will have the form $L(\lambda^\omega, \mathcal{H})$ where
   1.1 $\mathcal{H} \subseteq \lambda$,
   1.2 $\lambda = \sup(\mathrm{Ord} \cap \mathcal{H})$ and
   1.3 $\lambda = (\Theta^+)^{L(\lambda^\omega, \mathcal{H})}$ (recall Θ is the successor of the continuum).
2. The forcing is $\mathbb{P}_{\max} * \mathrm{Add}(\omega_3, 1) * \mathrm{Add}(\omega_4, 1)$.

The plan: the forcing part

The forcing portion reduces to forcing the Axiom of Choice over the Chang model:

Open Problem: Assume large cardinals and let $C = L(Ord^\omega)$. Is there a (class) forcing extension W of C such that $C = (L(Ord^\omega))^W$ and $W \models \mathrm{ZFC}$.

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By a result of Woodin, large cardinals imply that the theory of the Chang model cannot be changed by forcing, so the answer is either yes or no.

The plan: inner model part

The argument goes as follows:

1. (First Step) Prove Steel's Hod Pair Capturing below a Woodin cardinals that is a limit of Woodin cardinals. It says that under AD^+ every set of reals is projectively definable from an iteration strategy of a hod mouse.

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2. (Second step) Show that the existence of divergent models of AD implies the existence of a hod mouse with a certain large cardinal configuration.
3. (Third step) Show that the derived model of the hod mouse from the second step is the desired ground model.

Some applications and further motivations

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We say (M, N) is a divergent pair if M and N are proper models of AD and there is $A \in \mathcal{P}(\mathbb{R}) \cap M$ and $B \in \mathcal{P}(\mathbb{R}) \cap N$ such that $A \notin N$ and $B \notin M$.

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Assume the existence of a Woodin cardinal that is a limit of Woodin cardinals. Then there is an inner model in which there is a divergent pair.

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Assume the existence of a Woodin cardinal that is a limit of Woodin cardinals. Then there is an inner model in which there is a divergent pair.

Thus, not all extensions of AD have a $\subseteq$-minimal model.

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Corollary

The theory $\mathrm{MM}^{++}(c) + \neg\Box_{\omega_3} + \neg\Box(\omega_3)$ *is consistency wise weaker than a Woodin cardinal that is a limit of Woodin cardinals.*

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Theorem (Jensen-Schimmerling-Schindler-Steel)

Assume $\mathsf{MM}^{++}(\mathfrak{c}) + \neg\Box_{\omega_3} + \neg\Box(\omega_3)$ *and suppose* K^c_{MiSc} *exists. Then* K^c_{MiSc} *has a superstrong cardinal.*

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This is pretty bad for inner model theory.

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3. Axioms of determinacy.

Background: Square principles.

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Given a cardinal κ, the principle $\square_\kappa$ says that there exists a sequence $(C_\alpha : \alpha < \kappa^+)$ such that for each $\alpha < \kappa^+$,

- each C_α is a closed cofinal subset of α;
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Definition

Given an ordinal γ, the principle $\square(\gamma)$ says that there exists a sequence $(C_\alpha : \alpha < \gamma)$ such that

- for each $\alpha < \gamma$,
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- there is no thread through the sequence, i.e., there is no closed unbounded $E \subseteq \gamma$ such that $C_\alpha = \alpha \cap E$ for every limit point α of E.

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PFA *implies that* $0^\#$ *exists (i.e., there is an elementary embedding* $j : L \to L$ *or equivalently for some singular cardinal* κ, $(\kappa^+)^L < \kappa^+$*.)*

The inner model program

Definition (The Constructible Universe)

Define $(L_\alpha : \alpha < Ord)$ by recursion as follows.

1. $L_0 = \emptyset$,
2. for $\alpha \in Ord$, $L_{\alpha+1} = \{A \subseteq L_\alpha : A$ is definable over $(L_\alpha, \in)$ with parameters$\}$,
3. for limit $\alpha \in Ord$, $L_\alpha = \cup_{\beta<\alpha} L_\beta$,
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Theorem (Scott, 1938)

There are no measurable cardinals in L.

The inner model program

The inner model program: Construct *canonical* inner models for large cardinals.

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(6) κ is a strong cardinal if it is λ-strong for all λ.

Woodin cardinals

Definition

δ is a Woodin cardinal if δ is an inaccessible cardinal and for all $f : \delta \to \delta$ there is $\kappa < \delta$ and $j : V \to M$ with the property that $crit(j) = \kappa$ and $V_{j(f)(\kappa)} \subseteq M$.

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Exercise

δ is a Woodin cardinal if δ is an inaccessible cardinal and for all $A \subseteq \delta$ there is $\kappa < \delta$ such that for all $\lambda < \delta$ there is $j : V \to M$ with the property that $crit(j) = \kappa$ and $A \cap V_\lambda = j(A) \cap V_\lambda$.

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$$E = \{(a, A) \in \lambda^{<\omega} \times M : a \in j(A)\}.$$

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Extenders can be defined formally but they all are derived from embeddings.

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$$E = \{(a, A) \in \lambda^{<\omega} \times M : a \in j(A)\}.$$

E is called the (κ, λ)-extender derived from j. It is an M-extender (i.e. measures sets in M).

Extenders can be defined formally but they all are derived from embeddings.

Large cardinals: λ-generated embeddings

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Suppose E is a (countably complete) (κ, λ)-extender. There is then an (M, λ)-generated pair (j, N) such that E is the (κ, λ)-extender derived from j.

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The (j, N) above is constructed via the usual ultrapower construction. We denote the pair by j_E and $Ult(M, E) =_{def} M_E$

Large cardinals: Short extenders vs Long extenders

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If E is short then $E_a =_{def} \{A : (a, A) \in E\}$ is an ultrafilter concentrating on $[\kappa]^{|a|}$.

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Inner model theory: premice and mice

A simple pre-mouse is a structure of the form $\mathcal{M} = (M, \mu)$ where

(1) M is a transitive model of ZFC $-$ Powerset,
(2) M has a largest cardinal κ,
(3) μ is an M-ultrafilter over κ,
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Given a simple pre-mouse one can develop a theory of iterated ultrapowers of $\mathcal{M}$. If all the models of this linear iteration are well-founded we say that $\mathcal{M}$ is iterable. A simple mouse is an iterable pre-mouse.

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Something to think about: Show that if the first ω_1-iterated ultrapowers of $\mathcal{M}$ are well-founded then all iterated ultrapowes of $\mathcal{M}$ are well-founded.

Inner model theory: premice and mice

The goal of inner model theory is to develop the theory of *minimal* canonical models for large cardinals.

A potential problem: Suppose $x \subseteq \omega$ is a highly non-definable real and E is an extender. Then $x \in L[\vec{E}]$ where $\vec{E}(n) = E$ if $n \in x$ and otherwise $\vec{E}(n) = \emptyset$. So we have to be careful which $\vec{E}$'s to pick.

Those that work are called good extender sequences.

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Definition

A premouse is a model of the form $L_\alpha[\vec{E}]$. A mouse is an iterable premouse.

General iterability is a more complicated notion than linear iterability.

Inner model theory: iterability

Suppose M is a premouse.

1. We can iterate M linearly by taking an extender $E \in M$, forming $Ult(M, E) =_{def} M_1$ and then taking $E_1 \in M_1$ and forming $Ult(M_1, E_1)$ and etc.

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2. When can we take E_1 and instead of forming $Ult(M_1, E_1)$, form $Ult(M, E_1)$?
3. We could do that if M has the same powerset of $crit(E_1)$ as M_1.

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5. A strategy is an α-strategy if it is a wining strategy for II in the iteration game that lasts α steps.

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A pair $(\mathcal{M}, \Sigma)$ is called a mouse pair if $\mathcal{M}$ is a mouse and Σ is an $\omega_1 + 1$-iteration strategy for $\mathcal{M}$.
We say $(\mathcal{N}, \Lambda)$ is a tail of $(\mathcal{M}, \Sigma)$ if $\mathcal{N}$ is a Σ-iterate of $\mathcal{M}$ obtained via iteration some iteration tree $\mathcal{T}$ and Λ is given by $\Lambda(\mathcal{U}) = \Sigma(\mathcal{T}^\frown\mathcal{U})$.

Inner model theory: comparison

Theorem (Mitchell-Steel, and others)

Suppose $(\mathcal{M}, \Sigma)$ *and* $(\mathcal{N}, \Lambda)$ *are two mouse pairs. There are then a mouse pairs* $(\mathcal{M}', \Sigma')$ *and* $(\mathcal{N}', \Lambda')$ *such that*
(1) $(\mathcal{M}', \Sigma')$ *is a tail of* $(\mathcal{M}, \Sigma)$*,*
(2) $(\mathcal{N}', \Lambda')$ *is a tail of* $(\mathcal{N}, \Lambda)$*,*
(3) either

- *for some* $\alpha \leq \mathrm{Ord} \cap \mathcal{N}'$, $\mathcal{M}' = L_\alpha[\vec{E}^{\mathcal{N}'}]$ *or*
- *for some* $\alpha \leq \mathrm{Ord} \cap \mathcal{M}'$, $\mathcal{N}' = L_\alpha[\vec{E}^{\mathcal{M}'}]$.

We usually write $\mathcal{M}' \trianglelefteq \mathcal{N}'$ to mean the first part of clause 3.

Inner model theory: K^c-constructions

We define a sequence $\vec{C} = (\mathcal{M}_\alpha, \mathcal{N}_\alpha, E_\alpha : \alpha \in \mathrm{Ord})$ as follows.

1. Set $\mathcal{M}_0 = \emptyset$.
2. Suppose $\alpha \in \mathrm{Ord}$ and "Add Extender" condition doesn't hold. Then set
$$\mathcal{N}_\alpha = L_1(\mathcal{M}_\alpha)$$
$$\mathcal{M}_{\alpha+1} = tc(Hull^{\mathcal{M}_\alpha}).$$
3. If α is limit and $(\mathcal{M}_\beta : \beta < \alpha)$ has been defined then set $\mathcal{M}_\alpha = lim_{\beta<\alpha}\mathcal{M}_\beta$.

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3. If α is limit and $(\mathcal{M}_\beta : \beta < \alpha)$ has been defined then set $\mathcal{M}_\alpha = lim_{\beta<\alpha}\mathcal{M}_\beta$.
4. More precisely, we define $\mathcal{M}_\alpha$ by induction on its cardinals. Given $\mathcal{M}_\alpha|\kappa$ set $\mathcal{M}_\alpha|(\kappa^+)^{\mathcal{M}_\alpha} = \mathcal{M}_\beta|(\kappa^+)^{\mathcal{M}_\beta}$ for the least β such that for all $\gamma \in [\beta, \alpha)$, $\mathcal{M}_\gamma|(\kappa^+)^{\mathcal{M}_\gamma} = \mathcal{M}_\beta|(\kappa^+)^{\mathcal{M}_\beta}$.

$\mathcal{M}_{\alpha+1}$ and the Add Extender Condition: the certification condition

Suppose there are two cardinals $\kappa < \lambda$ and F be a (κ, λ)-extender that is *certified* and set

$$E_\alpha = F \cap \mathcal{M}_\alpha,$$
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K^c_{MiSch} was used by Jensen and Steel to remove the measurable from the theory of the core model (which was based on Steel's K^c. This work was awarded the second Hausdorff Medal.

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Inner model theory: The Iterability conjecture

The Iterability conjecture:
Suppose N is a model appearing on a K^c-construction and $\pi : M \to N$ is some countable hull of N. Then M has an $\omega_1 + 1$ iteration strategy.

The Iterability conjecture implies that K^c converges (due to many people).

Corollary

It is consistent relative to a Woodin cardinal that is a limit of Woodin cardinals that the iterability conjecture for K^c_{MiSch} *is false.*

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Inner Model Theory: Road Map

1. Itay Neeman, Determinacy in $L(\mathbb{R})$, a chapter for the Handbook of Set Theory (Foreman, Kanamori, Magidor, eds.), Springer-Verlag 2009.
2. Ralf Schindler and Martin Zeman, Fine structure theory, a chapter for the Handbook of Set Theory (Foreman, Kanamori, Magidor, eds.), Springer-Verlag 2009.
3. John Steel, Outline of Inner Model Theory, a chapter for the Handbook of Set Theory (Foreman, Kanamori, Magidor, eds.), Springer-Verlag 2009.

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(1) The proof is via the Derived Model Theorem which is the topic of the last tutorial.
(2) Woodin also demonstrated that large cardinals imply that the minimal model of $AD_{\mathbb{R}}$ exists.

The Axiom of Determinacy: Extensions

Suslin cardinals and Suslin sets are important objects that descriptive set theorists study under AD.

Definition

We say $A \subseteq \mathbb{R} = \omega^\omega$ is κ-Suslin if there is a tree $T \subseteq \cup_{n<\omega}\omega^n \times \kappa^n$ such that $A = p[T]$. κ is called a Suslin cardinal if there is a κ-Suslin set of reals which is not λ-Sulsin for all $\lambda < \kappa$.

AD^+ is the theory that (essentially) says that the portion of the universe coded by Suslin, co-Suslin sets is Σ_1-elementary in the universe.

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$\Theta = \sup\{\alpha : \text{there is a surjection } f : \mathbb{R} \to \alpha\}$. Θ_{reg} is the theory $ZF + AD_{\mathbb{R}} +$ "Θ is a regular cardinal".

Θ_{reg} implies AD^+.

Theorem

Suppose there is a Woodin cardinal that is a limit of Woodin cardinals. Then the minimal model of Θ_{reg} exists.

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Major Open Problem: Does AD imply AD^+.

$AD^{L(\mathbb{R})}$ implies $(AD^+)^{L(\mathbb{R})}$.$AD_{\mathbb{R}} + DC$ implies AD^+.

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Theorem

Suppose there is a Woodin cardinal that is a limit of Woodin cardinals. Then the minimal model of Θ_{reg} *exists.*

The Axiom of Determinacy: The Largest Suslin Axiom

Definition

The Largest Suslin Axiom is the conjunction of the following statements:

1. ZF $+$ AD$^+$.
2. There is a largest Suslin cardinal κ.
3. The largest Suslin cardinal is inaccessible with respect to OD surjections, i.e., for all $\gamma < \kappa$ there is no OD surjection $f : \mathcal{P}(\gamma) \to \kappa$.

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Theorem

Suppose there is a Woodin cardinal that is a limit of Woodin cardinals. Then the minimal model of LSA *exists.*

The Axiom of Determinacy: The Solovay sequence

Definition

The Solovay Sequence is a closed sequence $(\theta_\beta : \beta \leq \eta)$ defined as follows.

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By requiring that η is large, we get a hierarchy of axioms. E.g. $\mathsf{AD}^+ + \Theta = \theta_0$ is a strictly weaker axiom than $\mathsf{AD}^+ + \Theta = \theta_1$.

The Axiom of Determinacy: LSA vs Θ_{reg}

Fact

Assume Θ_{reg}. Then for every $\alpha < \Theta$ there is $A \subseteq \mathbb{R}$ such that $L(A, \mathbb{R}) \vDash \theta_{\alpha+1} = \Theta$.

The Axiom of Determinacy: LSA vs Θ_{reg}

Fact

Assume Θ_{reg}. *Then for every* $\alpha < \Theta$ *there is* $A \subseteq \mathbb{R}$ *such that* $L(A, \mathbb{R}) \models \theta_{\alpha+1} = \Theta$.

Fact

Assume LSA. *Let* κ *be the largest Suslin cardinal. Then* κ *is a member of the Solovay sequence and moreover, if* Δ_κ *is the collection of those sets of reals whose Wadge rank is* $< \kappa$ *then* $L(\Delta_\kappa) \models \Theta_{\text{reg}}$.

The Axiom of Determinacy: Road Map

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