Rigidity of the Endomorphisms of the Calkin algebra

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Liftings

Fix two Borel structures M, N in a language $\mathcal{L}$, i.e. structures whose universes are Polish spaces and where all the interpretations of functions and relations are Borel. Let E (resp. F) be an equivalence relation on M (resp. N) such that M/E (resp. N/F) is again a Borel structure in $\mathcal{L}$.

Let $\Phi\colon M/E \to N/F$ be a Borel homomorphism. A **lifting** $\Phi_*\colon M \to N$ is a map making the following diagram commute

$$\begin{array}{ccc} M & \xrightarrow{\Phi_*} & N \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_F} \\ M/E & \xrightarrow{\Phi} & N/F \end{array}$$

- Φ is **topologically trivial** if it has a Borel measurable lifting.
- Φ is **(algebraically) trivial** if it has a lifting preserving as much algebraic structure as possible.

The Rigidity Question

Question

Under what assumptions can we say that every isomorphism (or homomorphism) between two quotient Borel structures M/E and N/F is topologically or algebraically trivial?

We can consider three types of assumptions

- Assumptions on the structures M and N.
- Assumptions on the equivalence relations E and F.
- Additional set-theoretic assumptions.

The Rigidity Conjecture

Conjecture Template

- *The Continuum Hypothesis (CH) implies that M/E has $2^{\mathfrak{c}}$ automorphisms (and therefore $2^{\mathfrak{c}}$ topologically nontrivial automorphisms).*
- *Forcing Axioms imply that every isomorphism between M/E and N/F is topologically trivial.*

There are several examples from the categories of Boolean algebras, topological spaces (Čech–Stone reminders) and C^*-algebras where the Conjecture holds. In many of these cases, **topological triviality** of an isomorphism automatically implies **algebraic triviality**.

The Calkin algebra

Let H denote the complex, separable, infinite-dimensional Hilbert space $\ell^2(\omega)$. Set

$$\mathcal{B}(H) = \{ T \colon H \to H \mid T \text{ is linear and continuous} \}.$$

Let $\mathcal{K}(H)$ be the ideal of all operators in $\mathcal{B}(H)$ which are norm-limit of finite-rank operators.

The **Calkin algebra** $\mathcal{Q}(H)$ is the quotient $\mathcal{B}(H)/\mathcal{K}(H)$.

An automorphism $\Phi \colon \mathcal{Q}(H) \to \mathcal{Q}(H)$ is (algebraically) **trivial** if and only if $\Phi(a) = vav^*$ for some unitary $v \in \mathcal{Q}(H)$. Equivalently, Φ is trivial if and only if there is a unitary $v \in \mathcal{Q}(H)$ such that $v\Phi(\cdot)v^*$ lifts to an isomorphism of $\mathcal{B}(H)$.

Under the Conitnuum Hypothesis

One normally uses CH to show that quotient structures have $2^{\mathfrak{c}}$ automorphisms, so not all of them can be topologically trivial (there are only $\mathfrak{c}$ topologically trivial automorphisms). This is often done with back-and-forth arguments using **saturation** or **homogeneity**:

- $\mathcal{P}(\omega)/\mathrm{Fin}$ (Rudin), and $\mathcal{P}(\omega)/\mathcal{I}$ where $\mathcal{I} \supseteq \mathrm{Fin}$ is F_σ or a generalized density ideal (Just-Krawczyk).
- Let X be a second countable locally compact noncompact zero-dimensional space. Then $\beta X \setminus X \cong \beta\omega \setminus \omega$ and it has $2^{\mathfrak{c}}$ automorphisms (Parovicenko).
- Reduced products $\prod A_n / \bigoplus A_n$ for metric structures (A_n) with density $\leq \mathfrak{c}$ have $2^{\mathfrak{c}}$ automorphisms (Farah-Shelah).

Under the Conitnuum Hypothesis

The Calkin algebra is nor saturated nor countably homogeneous, but under CH it has $2^{\mathfrak{c}}$ automorphisms.

- **Phillips–Weaver (2007):** stratify $\mathcal{Q}(H) = \bigcup_{\alpha<\aleph_1} A_\alpha$, where $\{A_\alpha\}_{\alpha<\aleph_1}$ is an increasing sequence of separable subalgebras of $\mathcal{Q}(H)$, and build a complete tree of height $\aleph_1$ of partial automorphisms induced by unitaries. The construction is *very* careful, and all unitaries along the same branch are chosen to be homotopic, in order to avoid getting stuck at limit stages.
- **Farah (2011):** stratify $\mathcal{Q}(H)$ as union of Banach spaces resembling reduced products indexed over $\mathrm{Part}(\omega)$, and again assign to each of these Banach spaces a unitary inducing a partial automorphism, in a coherent way. If there is a cofinal subset of $\mathrm{Part}(\omega)$ of size $\aleph_1$ (i.e. if $\mathfrak{d} = \aleph_1$), then one can build $2^{\aleph_1}$ coherent distinct families of unitaries, each one giving a different automorphism of $\mathcal{Q}(H)$. If $2^{\aleph_1} > \mathfrak{c}$, this gives topologically nontrival automorphisms.

Under Forcing Axioms

- **Shelah (1982):** Consistently with ZFC, all automorphisms of $\mathcal{P}(\omega)/\mathrm{Fin}$ are trivial.
  Starting from a model of CH, perform a finite-support iteration of length $\aleph_2$ killing all potential non-trivial automorphisms of $\mathcal{P}(\omega)/\mathrm{Fin}$ (with $\diamondsuit_{\aleph_2}$ helping to intercept all potential non-trivial automorphisms throughout the iteration). The *killing* part boils down to a gap-preservation argument.
- **Shelah–Steprans (1989):** Combining the above with Kunen's freezing-gap technique, they proved that PFA implies that all automorphisms of $\mathcal{P}(\omega)/\mathrm{Fin}$ are trivial.
- **Velickovic (1992):** the same statement follows from OCA + $\mathrm{MA}_{\aleph_1}$.
- **Farah (2011):** Assume OCA. Then all automorphisms of $\mathcal{Q}(H)$ are trivial (i.e. inner).

Endomorphisms of the Calkin algebra

A unital endomorphism $\varphi : \mathcal{Q}(H) \to \mathcal{Q}(H)$ is **trivial** if there is a unitary $v \in \mathcal{Q}(H)$ such that $v\varphi(\cdot)v^*$ is liftable to a strong-strong continuous, unital $*$-homomorphism Φ of $\mathcal{B}(H)$ into $\mathcal{B}(H)$.

$$\begin{array}{ccc} \mathcal{B}(H) & \xrightarrow{\Phi} & \mathcal{B}(H) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{Q}(H) & \xrightarrow{v\varphi(\cdot)v^*} & \mathcal{Q}(H) \end{array}$$

Theorem (V., 2021)

Assume OCA. All unital endomorphisms of $\mathcal{Q}(H)$ are trivial.

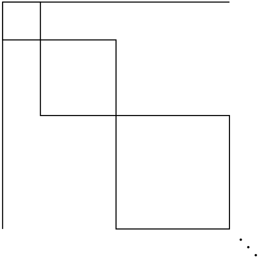
This result is contrast with the commutative world:

Theorem (Dow, 2014)

There exists a nontrivial copy of the Stone-Cěch remainder $\beta\omega \setminus \omega$ inside $\beta\omega \setminus \omega$.

Roadmap for a proof

Fix an endomorphism $\Phi\colon \mathcal{Q}(H) \to \mathcal{Q}(H)$. The work on $\mathcal{Q}(H)$ is first done locally on $\mathcal{D}[E]$, for $E \in \mathrm{Part}(\omega)$, whose elements look like



- Assume OCA. The restriction of Φ to $\mathcal{D}[E]$ is topologically trivial.
- If the restriction of Φ to $\mathcal{D}[E]$ is topologically trivial, then it is also (algebraically) trivial.
- Assume OCA. If the restriction of Φ to each $\mathcal{D}[E]$ is trivial, then Φ is trivial on $\mathcal{Q}(H)$.

Classification of Endomorphisms

Theorem (V., 2021)

Assume OCA, and let $End(\mathcal{Q}(H))$ be the set unital endomorphisms of $\mathcal{Q}(H)$, and $\sim_u$ the unitary equivalence. The map

$$\begin{aligned} Ind : End(\mathcal{Q}(H))/\sim_u &\to \mathbb{N} \setminus \{0\} \\ \varphi &\mapsto -ind(\varphi(q(S))) \end{aligned}$$

is a bijection. In particular, $End(\mathcal{Q}(H))/\sim_u$ is completely classified by $\mathbb{N} \setminus \{0\}$.

Corollary

Assume OCA. There is an isomorphism of semigroup between $(End(\mathcal{Q}(H))/\sim_u, \oplus)$ and $(\mathbb{N}^+, +)$, and between $(End(\mathcal{Q}(H))/\sim_u, \circ)$ and $(\mathbb{N}^+, \cdot)$.

All this badly fails under CH.

Subalgebras of the Calkin algebra

Theorem (Farah-Hirshberg-Vignati, 2018)

Every C^*-algebras A of density character $\aleph_1$ embeds into $\mathcal{Q}(H)$. In particular if CH holds, the Calkin algebra is $\mathfrak{c}$-universal.

Let $\mathcal{E} = \{C^*\text{-algebras that embed into } \mathcal{Q}(H)\}$. Under CH the family $\mathcal{E}$ is closed under most reasonable operations, such as tensor products and countable inductive limits.

Theorem (V., 2021)

Assume OCA:

- *If A is infinite-dimensional C^*-algebra, then $A \otimes_\gamma \mathcal{Q}(H) \notin \mathcal{E}$. In particular $\mathcal{Q}(H) \otimes_\gamma \mathcal{Q}(H) \notin \mathcal{E}$, hence $\mathcal{E}$ is not closed under tensor product.*
- *There exists an increasing sequence $A_n \in \mathcal{E}$ such that $\overline{\bigcup_n A_n} \notin \mathcal{E}$.*

In particular $\mathcal{Q}(H)$ is not $\mathfrak{c}$-universal.

Essential Bibliography

- *Corona Rigidity*, joint with Ilijas Farah, Saeed Ghasemi and Alessandro Vignati, arXiv:2201.11618 (2022), `https://arxiv.org/abs/2201.11618`.
- *Trivial Endomorphisms of the Calkin algebra*, Israel Journal of Mathematics 247 (2021), 873–903, `https://doi.org/10.1007/s11856-021-2284-0`.