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Reverse Mathematics and Dimension of Posets

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Contributions

This is joint work with Marta Fiori Carones and Alberto Marcone.

Outline

Introduction to Reverse Mathematics

Reverse Mathematics Goal

Reverse Mathematics is concerned with the following main question: which set existence axioms are needed to prove the theorems of ordinary mathematics?

The idea of Reverse Mathematics leans on this observation of Friedman: "When the theorem is proved from the right axioms, the axioms can be proved from the theorem".

Second Order Arithmetic

Research in Reverse Mathematics is carried out in the context of subsystems of Second Order Arithmetic.

The language of second order arithmetic $\mathcal{L}_2$ is a two-sorted language: there is a sort for number variables and a sort for set variables and the following extralogical symbols:

- constants $0, 1$;
- binary functions $+, \cdot$;
- binary relations $=, <$ between numbers and $\in$ between a number and a set

Arithmetical Hierarchy

Definition

The arithmetical hierarchy is the following classification of $\mathcal{L}_2$-formulas without set quantifiers.

- A formula with only bounded number quantifiers is Σ^0_0 or Π^0_0.
- If φ is Σ^0_n for some n, then the formula $\forall \vec{x} \varphi$ is Π^0_{n+1}.
- If φ is Π^0_n for some n, then the formula $\exists \vec{x} \varphi$ is Σ^0_{n+1}.
- If φ is Σ^0_n and Π^0_n for some n, then the formula is Δ^0_n.

Set Existence, Bounding and Induction

Definition

For each family Γ of $\mathcal{L}_2$-formulas, $\mathrm{I}\Gamma$ is the scheme of axioms

$$(\varphi(0) \wedge \forall x\,(\varphi(x) \to \varphi(x+1))) \to \forall x\,(\varphi(x))$$

where $\varphi(x) \in \Gamma$.
On the other hand, Γ-CA is the scheme of axioms

$$\exists X \forall x (x \in X \leftrightarrow \varphi(x))$$

where X is not free in $\varphi(x)$ and $\varphi(x) \in \Gamma$.

Full Second Order Arithmetic

Full second order arithmetic is the theory with algebraic axioms for the basic properties of natural numbers, $\mathrm{I}\Gamma$ and Γ-CA where Γ coincides with the set of all $\mathcal{L}_2$-formulas. Such theory is denoted by Z_2.

Subsystems of second order arithmetic are built weakening the induction and/or comprehension axioms of Z_2.

The systematic search for the subsystems of second order arithmetic which are sufficient and necessary to prove theorems of ordinary mathematics, was started by Harvey Friedman around 1970 and pursued by Stephen Simpson and many others.

Big Five

The most important subsystems are known as the big five:

1. RCA_0: basic algebraic axioms, $\mathrm{I}\Sigma^0_1$ and Δ^0_1-CA;
2. WKL_0: RCA_0 + Weak König's Lemma;
3. ACA_0: RCA_0 + comprehension for all arithmetical formulas;
4. ATR_0: ACA_0 + arithmetical transfinite recursion;
5. Π^1_1-CA_0: RCA_0 + Π^1_1-comprehension.

Zoo of Reverse Mathematics

In 1995 Seetapun showed that Ramsey Theorem for pairs and two colors RT^2_2 does not imply ACA_0. It was already known that WKL_0 does not prove RT^2_2. In 2012 Liu showed that RT^2_2 does not imply WKL_0.

Afterwards, many statements provable in ACA_0, unprovable in RCA_0, and not equivalent to either ACA_0 or WKL_0, have been discovered.

This phenomenon became known as the Zoo of Reverse Mathematics.

Introduction to Dimension of Posets

Posets

A poset is a set X together with a binary relation $\prec$ transitive and irreflexive. Sometimes the relation is asked to be transitive, reflexive and antisymmetric.

Let $x \mid y$ if and only if $x \neq y$, $x \not\prec y$ and $y \not\prec x$.

A linear order is a poset in which any two distinct elements are comparable. A linearly ordered set is also called a chain.

Dimension

Definition

Let $(X, \prec)$ be a poset and let L be a family of linearizations. We say that L realize $(X, \prec)$ if $\bigcap L = \prec$. The dimension of $(X, \prec)$ is the least cardinality of a realization of $(X, \prec)$.

A chain has dimension 1 while an antichain has dimension 2.

We will focus on posets of finite dimension. In fact, when we deal with subsystems of second order arithmetic, we work with countable posets whose dimension is at most countable. Since we are interested in upper bound theorems, the finite dimensional case is the interesting one.

Dimension always exists in RCA_0

Theorem (RCA_0) [Szpilrajn]

Every poset can be extended to a linear order.

Proposition (RCA_0)

For each $(X, \prec)$, there exists a set $\{\lhd_n : n \in \mathbb{N}\}$ of linearizations such that for each $x, y \in X$ $x \prec y$ if and only if $x \lhd_n y$ for every $n \in \mathbb{N}$.

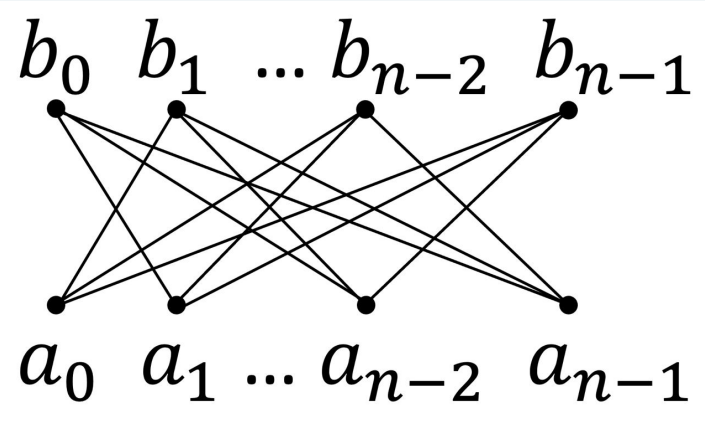
Thus in RCA_0 any poset has a dimension.

The poset F_n

One may wonder if, for any n, there is a poset of dimension n. This is the case: the poset below has indeed dimension n.

Definition

Let $1 < n \in \mathbb{N}$ and $F_n = \{a_i, b_i : i < n\}$. We equip F_n with the relation $a_i \prec b_j$ whenever $i \neq j$.



For simplicity, we will refer to such poset simply by F_n.

Bounding Theorems

We are interested in studying the strength of some bounding theorems from the point of view of Reverse Mathematics.

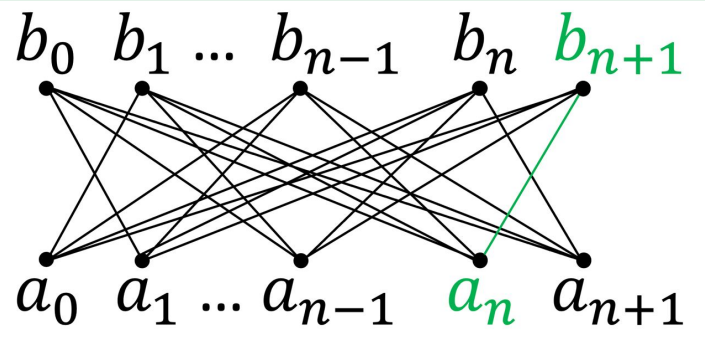
- DB_p: for each $(X, \prec)$ and each $x_0 \in X$, $\dim(X, \prec) \leq \dim(X \setminus \{x_0\}, \prec) + 1$.
- DBc_n: fixed $n > 0$, for each $(X, \prec)$ and each pairwise incomparable and disjoint chains $C_i \subseteq X$ for $i < n$, $\dim(X, \prec) \leq \dim\left(X \setminus \left(\bigcup_{i<n} C_i\right), \prec\right) + \max\{2, n\}$.

The abnormal case is DBc_1.

DBc_1 is optimal

Example

Consider F_{n+2} and the chain $C = \{a_n, b_{n+1}\}$. The poset $F_{n+2} \setminus C$ is a copy of F_{n+1} with the extra relation $a_n \prec b_n$.

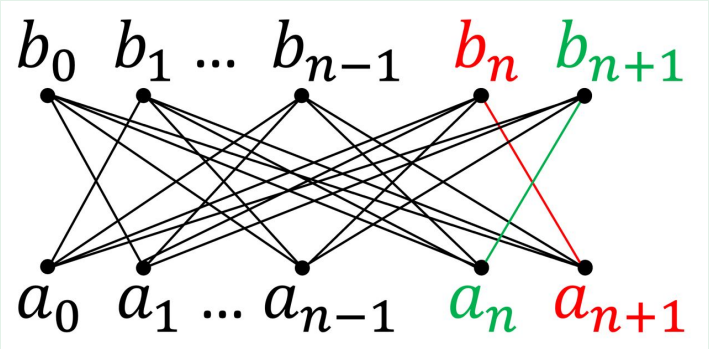


It is provable that the dimension of $F_{n+2} \setminus C$ is n. Hence $\dim(F_{n+2} \setminus C) = n = \dim(F_{n+2}) - 2$.

DBc_2 is optimal

Example

Consider F_{n+2}, $C_0 = \{a_{n+1}, b_n\}$ and $C_1 = \{a_n, b_{n+1}\}$. The poset $F_{n+2} \setminus (C_0 \cup C_1)$ is exactly F_n so its dimension is n.



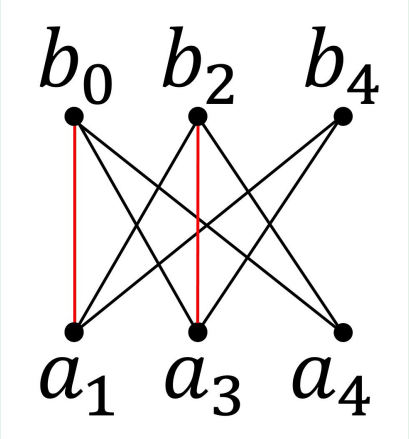
Therefore $\dim(F_{n+2} \setminus (C_0 \cup C_1)) = n = \dim(F_{n+2}) - 2$.

DBc_2 chains must be incomparable

Do the chains in DBc_2 really have to be incomparable?

Example

Consider F_5, $C_0 = \{a_0, b_1\}$ and $C_1 = \{a_2, b_3\}$. The poset $F_5 \setminus (C_0 \cup C_1)$ looks like F_3 plus two extra comparabilities.



The poset $F_5 \setminus (C_0 \cup C_1)$ has dimension 2. Therefore we have that $\dim(F_5 \setminus (C_0 \cup C_1)) = 2 = \dim(F_5) - 3$.

A weaker Bound

We can state a weaker bound without the requirement of incomparability of the chains.

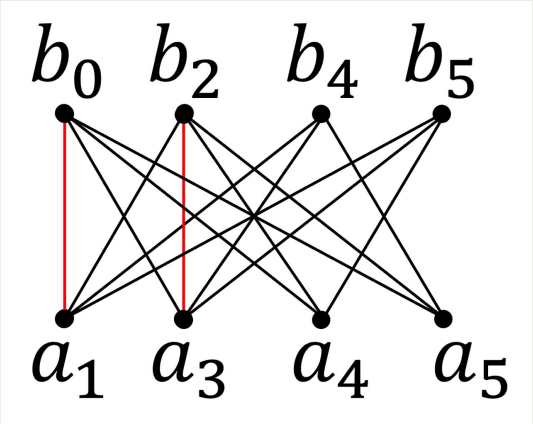
Fixed n for each poset $(X, \prec)$ and each family of pairwise disjoint chains $C_i \subseteq X$ for $i < n$,
$\dim(X, \prec) \leq \dim(X \setminus (\bigcup_{i<n} C_i), \prec) + 2n$.

Notice that such bound is obtained applying n times DBc_1.

The weak Bound is optimal

Example

Consider F_6, $C_0 = \{a_0, b_1\}$ and $C_1 = \{a_2, b_3\}$. The poset $F_6 \setminus (C_0 \cup C_1)$ looks like F_4 plus two extra comparabilities.

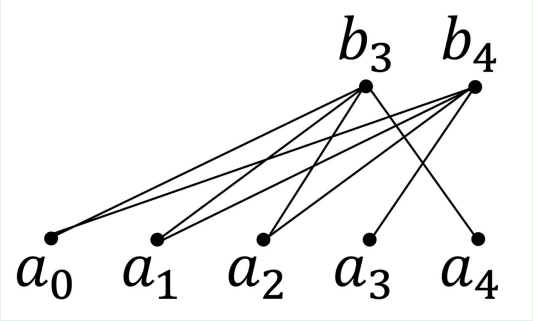


The poset $F_6 \setminus (C_0 \cup C_1)$ has dimension 2. Therefore we have that $\dim(F_6 \setminus (C_0 \cup C_1)) = 2 = \dim(F_6) - 4$.

DBc_n is optimal

Example

Consider F_5, $C_0 = \{b_0\}$, $C_1 = \{b_1\}$ and $C_2 = \{b_2\}$. The poset $F_5 \setminus (C_0 \cup C_1 \cup C_2)$ has the following form:



It can be proved that the dimension of $F_5 \setminus (C_0 \cup C_1 \cup C_2)$ is 2. Therefore $\dim(F_5 \setminus (C_0 \cup C_1 \cup C_2)) = 2 = \dim(F_5) - 3$.

The previous example can be modified to prove that DBc_n is optimal: it suffices to consider F_{n+2}.

Reverse Mathematics of Bounding Theorems

Preliminary Lemmas

Lemma (RCA_0) [Cholak, Marcone, Solomon]

The following are equivalent:

1. WKL_0;
2. every acyclic relation extends to a partial order;
3. every acyclic relation extends to a linear order.

Lemma (WKL_0)

Let $(X, \prec)$ and let $C_0, C_1 \subseteq X$ be incomparable and disjoint chains. There exists a linear order $(X, \lhd)$ extending $(X, \prec)$ such that for each $x \in X$, $c_0 \in C_0$ and $c_1 \in C_1$, if $x \mid c_0$ then $x \lhd c_0$ while if $x \mid c_1$ then $c_1 \lhd x$.

For each n $\mathsf{WKL}_0 \vdash DBc_n$

Using the previous Lemmas it can be shown that, for $n \geq 2$, WKL_0 proves DBc_n. DBc_1 is obtained by DBc_2 considering an empty chain.

Next we want to prove our main result: that WKL_0 is necessary for DBc_1 and DBc_2.

Main Theorem

Theorem (RCA_0) [Marcone, V.]

The following are equivalent:

1. WKL_0;
2. for each $(X, \prec)$ and each $C_0, C_1 \subseteq$ incomparable and disjoint chains,
$$\dim(X, \prec) \leq \dim(X \setminus (C_0 \cup C_1), \prec) + 2;$$
3. for each $(X, \prec)$ and each chain $C \subseteq X$,
$$\dim(X, \prec) \leq \dim(X \setminus C), \prec) + 2.$$

The interesting direction is $3 \implies 1$.

$3 \implies 1$ Sketch of Proof 1

Let f, g be 1-1 functions with disjoint ranges: we want to show that there exists a separator set A. Let

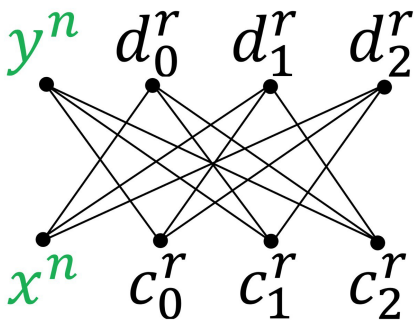
$$X = \{x^i, y^i : i \in \mathbb{N}\} \cup \{c_j^r, d_j^r : j < 3, r \in \mathbb{N}\} \cup \{p_j^s, q_j^s : j < 3, s \in \mathbb{N}\}.$$

We define a partial order $\prec$ on X. For each $n \in \mathbb{N}$ we construct a layer X_n which, if $n \notin \operatorname{ran}(f) \cup \operatorname{ran}(g)$, is an antichain consisting of x^n and y^n.

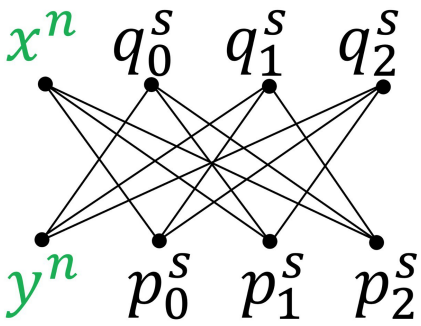
Otherwise...

Sketch of Proof 2

.. if $f(r) = n$ then X_n is a copy of F_4 with $x^n = a_0$ and $y^n = b_0$

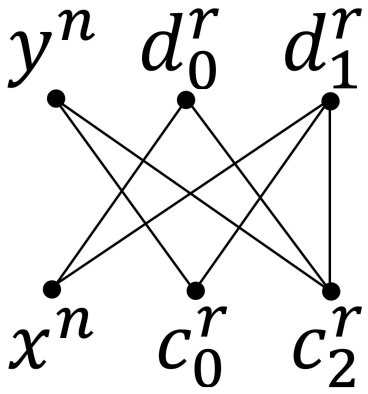


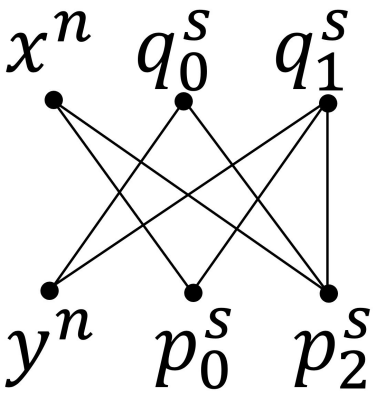
and if $g(s) = n$ then X_n is a copy of F_4 with $y^n = a_0$ and $x^n = b_0$.



Sketch of Proof 3

Let $C = \{c_1^r, d_2^r, p_1^s, q_2^s : s, r \in \mathbb{N}\}$ which is a chain.
We consider the poset $(X \setminus C, \prec)$. If $n \in \mathrm{ran}(f) \cup \mathrm{ran}(g)$ then $X_n \setminus C$ has one of the following forms:





It can be proved that $\dim(X \setminus C, \prec) = 2$.

Technical Lemma on F_n

By 3 $\dim(X, \prec) \leq \dim(X \setminus C, \prec) + 2 = 4$.

We fix a realization $\lhd_0^*, \lhd_1^*, \lhd_2^*, \lhd_3^*$ of $(X, \prec)$. For each n, $\lhd_0^* \restriction_{X_n}, \lhd_1^* \restriction_{X_n}, \lhd_2^* \restriction_{X_n}, \lhd_3^* \restriction_{X_n}$ realize $(X_n, \prec)$.

The next Lemma is useful.

Lemma (RCA_0)

For each set of linearizations $\lhd_0, \ldots, \lhd_{n-1}$ that realizes F_n and each $k < n$ there exists exactly one i such that $b_k \lhd_i a_k$.

Conclusion of the Proof

The previous Lemma ensures us that if $n \in \mathrm{ran}(f)$ then $|\{i < 4 : y^n \lhd_i^* x^n\}| = 1$ and if $n \in \mathrm{ran}(g)$ then $|\{i < 4 : y^n \lhd_i^* x^n\}| = 3$. If $n \notin \mathrm{ran}(f) \cup \mathrm{ran}(g)$ we only know that $1 \leq |\{i < 4 : y^n \lhd_i^* x^n\}| \leq 3$.

For each n, we say that $n \in A$ if and only if $|\{i < 4 : y^n \lhd_i^* x^n\}| = 1$. We notice that A separates $\mathrm{ran}(f)$ and $\mathrm{ran}(g)$ and is Σ^0_0 definable, so RCA_0 proves its existence.

Further Work

Preliminary Lemma

We recall DB_p: for each poset $(X, \prec)$ and each $x_0 \in X$, $\dim(X, \prec) \leq \dim(X \setminus \{x_0\}, \prec) + 1$.

The next Lemma is called the Linear Separation Lemma (LSL).

Lemma (RCA_0) [Fiori Carones, Marcone]

Let $(X, \lhd)$ be a linear order and let $I, F \subseteq X$ such that for each $i \in I$ and each $f \in F$, $i \lhd f$. Then there exists an initial interval $B \subseteq X$ such that $I \subseteq B$ and $F \cap B = \emptyset$.

An Attempt for DB_p

Fix $n_0 \in \omega$.

Theorem (RCA_0) [Marcone, V.]

For each $(X, \prec)$ and each $x_0 \in X$, if $\dim(X \setminus \{x_0\}, \prec) = n_0$ then $\dim(X, \prec) \leq n_0 + 1$.

We fix a realization $\lhd_0, \ldots, \lhd_{n_0-1}$ of $(X \setminus \{x_0\}, \prec)$ and we define the Σ^0_0 sets $I = \{x : x \in X \wedge x \prec x_0\}$ and $F = \{x : x \in X \wedge x_0 \prec x\}$.

Thanks to *LSL*, for $0 < i < n_0$ we extend $\lhd_i$ to X, while we use $\lhd_0$ to generate two linearizations of $(X, \prec)$. Finally it is proved that these $n_0 + 1$ linearizations realize $(X, \prec)$.

Next Ideas

Since the proof of *LSL* is not uniform, we cannot use it an arbitrary number of times. Hence the previous proof works only for a fixed n_0 and does not apply to any poset.

Fiori Carones and Marcone showed that DB_p is provable both in WKL_0 and in $\mathrm{I}\Sigma^0_3$ and recently we improved the result showing that $\mathrm{I}\Sigma^0_2$ suffices. Our conjecture is that DB_p is equivalent to the disjunction $\mathsf{WKL}_0 \vee \mathrm{I}\Sigma^0_2$.

Another open questions that we are currently approaching is whether for each n DBc_n implies WKL_0. Recently we solved the case DBc_3 but we are still working on the other cases.

Thank you for your attention!