Set Theory vs. Topology

Foundations of Arithmetic & non-Archimedean Geometry

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What this talk is about

What is a **space**? The classical answer is given by point-set topology, and this has had a foundational influence on how we formulate key ideas in many areas of maths.

This talk will introduce some unique perspectives from point-free topology, and illustrate its differences with point-set topology by drawing upon two examples from recent work, partially joint w/ Steve Vickers: one relating to **Berkovich geometry**, the other relating to **arithmetic geometry**.

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This talk will introduce some unique perspectives from point-free topology, and illustrate its differences with point-set topology by drawing upon two examples from recent work, partially joint w/ Steve Vickers: one relating to **Berkovich geometry**, the other relating to **arithmetic geometry**.

Both results are surprising in different ways, but they highlight subtle interactions between topology & algebra that were previously obscured by the underlying set theory.

Foundations in Berkovich Geometry

Let $(K, |\cdot|)$ be a complete valued field, and $K[T]$ be the polynomial ring.

Multiplicative Seminorm

A **multiplicative seminorm** on D extending the norm of K is a map

$$|\cdot|_x \colon K[T] \to \mathbb{R}_{\geq 0}$$

satisfying the following:

- $|f+g|_x \leq |f|_x + |g|_x \qquad \forall, f, g \in K[T]$
- $|fg|_x = |f|_x|g|_x \qquad \forall, f, g \in K[T]$
- $|a|_x = |a| \qquad \forall a \in K$

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Berkovich Affine Line

The **Berkovich Affine Line** $\mathbb{A}^1_{\mathrm{Berk}}$ is a space defined as follows:

- *Underlying set of* $\mathbb{A}^1_{\mathrm{Berk}}$ = set of multiplicative seminorms on $K[T]$.
- *Topology of* $\mathbb{A}^1_{\mathrm{Berk}}$ = the Gel'fand topology, i.e. weakest topology such that all maps of the form

$$\begin{aligned} \psi_f \colon \mathbb{A}^1_{\mathrm{Berk}} &\longrightarrow \mathbb{R}_{\geq 0} \\ |\cdot|_x &\longmapsto |f|_x \end{aligned}$$

are continuous, for any $f \in K[T]$.

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Classifying Points of Berkovich spaces

Let $(K, |\cdot|)$ be a complete non-Archimedean valued field that is algebraically closed. A rigid disc is a subset $D_r(k) \subset K$ of the form

$$D_r(k) := \{b \in K \,\big|\, |b - k| \leq r\}.$$

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Berkovich's Classification Theorem

Suppose K is non-trivially valued. <u>Then</u> every point $|\cdot|_x \in \mathbb{A}^1_{\mathrm{Berk}}$ corresponds to a nested descending sequence of discs

$$D_{r_1}(k_1) \supseteq D_{r_2}(k_2) \supseteq \ldots \tag{1}$$

in the sense that

$$|\cdot|_x = \lim_{n\to\infty} |\cdot|_{D_{r_i}(k_i)} \tag{2}$$

where $|\cdot|_{D_r(k)}$ is the multiplicative seminorm canonically associated to $D_r(k)$.

Classifying Points of Berkovich spaces

The same construction and result holds for other rings as well. Here's another important example:

- Let $(K, |\cdot|)$ be a complete non-Arch. field that is algebraically closed.
- Denote $\mathcal{A} := K\{R^{-1}T\}$ to be ring of power series converging in radius R.
- Denote $\mathcal{M}(\mathcal{A})$ to be the analogous space of multiplicative seminorms on $\mathcal{A}$.

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Berkovich's Classification Theorem

Suppose K is non-trivially valued. Then, every point $|\cdot|_x \in \mathcal{M}(\mathcal{A})$ is approximated by a nested descending sequence of discs

$$D_{r_1}(k_1) \supseteq D_{r_2}(k_2) \supseteq \ldots \tag{3}$$

in the same sense as before.

On the hypothesis of "non-trivially valued"

The space of multiplicative seminorms is still well-defined even when K is trivially valued.[1]

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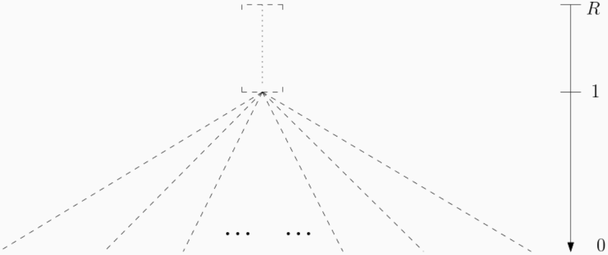
The space of multiplicative seminorms is still well-defined even when K is trivially valued.[1] In fact, Berkovich [Ber90] gives an explicit description of these spaces, depending on whether the radius of convergence $R < 1$ or $R \geq 1$.



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> *The second assumption [that K is non-trivially valued] is necessary [...] if the norm on K is trivial, then there are too few discs.*
>
> — **Jonsson [Jon15]**

Let us redefine the notion of rigid discs:

Formal Ball

Denote:

- $K_R := \{k \in K \mid |k| \leq R\}$ for some positive real $R > 0$
- Q_+ to be the set of positive rationals.

A **formal ball** is an element $(k, q) \in K_R \times Q_+$. We shall represent this more suggestively as $B_q(k)$. In particular, we write:

$$B_{q'}(k') \subseteq B_q(k) \text{ just in case } |k - k'| < q \wedge q' \leq q.$$

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Key Observation #1: Unlike rigid discs, the radius of formal balls are well-defined, i.e. $B_{q'}(k) = B_q(k')$ implies $q' = q$.

Perspective from Point-free Topology

Also, instead of working with nested sequences of rigid discs, let us consider:

R-good Filter

A **filter** $\mathcal{F}$ of formal balls is an inhabited subset of $K_R \times Q_+$ that is:

- Upward closed w.r.t $\subseteq$
- Closed under pairwise intersections.

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We call $\mathcal{F}$ an R-***good*** **filter** if it also satisfies:

- For any $k \in K_R$, and $q \in Q_+$ such that $R < q$, $B_q(k) \in \mathcal{F}$.
- If $B_q(k) \in \mathcal{F}$, there exists $B_{q'}(k') \in \mathcal{F}$ such that $q' < q$.

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Key Observation #2: Given an R-good filter $\mathcal{F}$, define $\mathrm{rad}_{\mathcal{F}} := \inf\{q \mid B_q(k) \in \mathcal{F}\}$ to be its *radius*. Notice $0 \leq \mathrm{rad}_{\mathcal{F}} \leq R$.

A Sharper Classification Theorem

Theorem (N.)

Setup:

- $(K, |\cdot|)$ is a complete non-Arch field that is algebraically closed – in particular, we allow K to be trivially-valued.
- $\mathcal{A} := K\{R^{-1}T\}$ is the ring of power series converging on radius R, and $\mathcal{M}(\mathcal{A})$ is the associated space of multiplicative seminorms.

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Then, the space of R-good filters is (classically) equivalent to $\mathcal{M}(\mathcal{A})$.

Slogan: The algebraic hypothesis of being non-trivially valued is in fact a point-set hypothesis.

We can now give new (and shorter) proofs of familiar charaterisations of Berkovich spectra:

Figure 1: LHS: $\mathcal{M}(\mathcal{A})$ when $R < 1$, RHS: $\mathcal{M}(\mathcal{A})$ when $R \geq 1$.

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- A topos can be regarded as a generalised space whose points are models of a geometric theory.

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- The language of formal balls reflect the localic perspective that it is the **opens** that are the basic units for defining a space, **<u>not</u>** the underlying **set** of points.
- A topos can be regarded as a generalised space whose points are models of a geometric theory. In particular, if the theory of multiplicative seminorms is *essentially propositional*, then we know the models also correspond to **completely prime filters**.

The View from Topos Theory

On Logic & Topology

> *Model theory rarely deals directly with topology; the great exception is the theory of o-minimal structures, where the topology arises naturally from an ordered structure.*
>
> — **E. Hrushovski and F. Loeser [HL16]**

> *While geometric logic can be treated as just another logic, it is an unusual one. [...] To put it another way, the geometric mathematics has an intrinsic continuity.*
>
> — **S. Vickers [Vic14]**

What is a space?

Point-set Topology

- Point = An element of a set
- Space = A set of points, along with a collection of opens satisfying specific properties ("topology").

Point-free Topology

- Point = A model of a geometric theory
- Space = The 'World' in which all models of the theory live ($\approx$ a topos)

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Geometric Logic

Let Σ be a (many-sorted) first-order signature (or vocabulary).

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- **Formula:** Let $\vec{x}$ be a finite vector of variables, each with a given sort. A *geometric formula* in context $\vec{x}$ is a formula built up using symbols from Σ via the following logical connectives: $=$, $\top$ (true), $\wedge$ (**finite** conjunction), $\bigvee$ (**arbitrary** disjunction), $\exists$.

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- **Theory:** A *geometric theory over* Σ is a set of axioms of the form

$$\forall \vec{x}.(\phi \to \psi),$$

where ϕ and ψ are geometric formulae.

Differences with classical logic

- Absence of negation $\neg$
- Allows for arbitrary (possibly infinite) disjunction

Special case: Propositional Theory

Suppose Σ is just a set of propositional symbols (in particular, no sorts).

- Geometric formulae are constructed from these symbols using $\top$, $\wedge$, $\bigvee$.
- A geometric theory over Σ is a set of axioms of the form $\phi \to \psi$.

Localic Space

Recall the following perspective from point-free topology.

- Point = A model of a geometric theory
- Space = The 'World' in which all models of the theory live

If the geometric theory is propositional, we call the corresponding space a **localic space**.

Example: Localic Reals

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Classical (finitary) logic: Dedekind cuts arise as *types* over the rationals $(\mathbb{Q}, <)$ considered as a dense linear order.

Geometric logic: Dedekind reals arise as *models* of a geometric theory.

Geometric Theory of Reals

The propositional theory $T_{\mathbb{R}}$ with propositional symbols $P_{q,r}$ (with $q, r \in \mathbb{Q}$, the rationals) and the axioms:

- $P_{q,r} \wedge P_{q',r'} \longleftrightarrow \bigvee \{P_{s,t} | \max(q, q') < s < t < \min(r, r')\}$
- $\top \longrightarrow \bigvee \{P_{q-\epsilon,q+\epsilon} | q \in \mathbb{Q}\}$ for any $0 < \epsilon \in \mathbb{Q}$.

Types and Models as Filters

Question: Given a point $x \in X$ in some space X, how does the family of open neighbourhoods containing x look like from its POV?

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Filter

For I an infinite set, $\mathcal{F} \subset \mathcal{P}(I)$ is a **filter** on I when:

(i) $A \subseteq B \subseteq I$ and $A \in \mathcal{F}$ implies $B \in \mathcal{F}$.

(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$

(iii) $I \in \mathcal{F}$.

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> **Type:** A partial type p over a model M corresponds to a filter on M for the *Boolean algebra* of M-definable subsets of M. If p is an **ultrafilter**, then we call p a (complete) type.
>
> **Model of (Geometric) Propositional Theory:** A propositional theory $\mathbb{T}$ can be associated to its *lattice* $L_{\mathbb{T}}$ of propositional formulae (modulo provable equality). The models of $\mathbb{T}$ are the **completely prime filters** of $L_{\mathbb{T}}$.

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Theory of Upper Reals

Consider a subset $R \subset \mathbb{Q}$. For suggestiveness, write "$R < r$" whenever $r \in R$.
Suppose R is subject to the axiom:

$$\forall r \in \mathbb{Q}.\left(R < r \longleftrightarrow \exists r' \in \mathbb{Q}.(R < r') \wedge (r' < r)\right)$$

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Remark: Morally speaking, an upper real R corresponds to the right Dedekind section of a real. More precisely, upper reals are *classically* equivalent the the usual Dedekind reals[2], but *intuitionistically* they are different.

[2] At least, once we ignore the infinities.

The Abstract vs. The Concrete

> *Category Theory is directed at the removal of the importance of a concrete construction. It provides a language to compare different concrecte constructions and in addition **provides a very new way to construct objects** [...] On the other hand, Model theory is concentrated on the gap between an abstract definition and a concrete construction.*
>
> — **David Kazhdan [Kaz06]**

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Point-free space = The 'World' in which all models of the theory live ($\approx$ a topos)

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$$R_M \subseteq M^{n_R}.$$

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Topos Theory. An **object** in a category (with formal properties similar to Set)[3], also equipped with interpretations of Σ.

[3] More precisely, a category satisfying Giraud's axioms, i.e. a topos.

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Topos Theory. An **object** in a category (with formal properties similar to Set), also equipped with interpretations of Σ. E.g. for an n-ary relation R on sorts $A_1 \times \cdots \times A_m$, a model defines a **subobject**

$$[[R]]_M \rightarrowtail [[A_1]]_M \times \cdots \times [[A_m]]_M$$

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Given a theory $\mathbb{T}$ over signature Σ, what are its models really?

Model Theory. Models come from sets.

Topos Theory. Models come from the category Set, but also other categories capable of interpreting logic . . . necessary to consider a wider range of models, because geometric logic is incomplete (unlike classical logic).

The Generic Model

In topos theory, we have access to a $\mathbb{T}$-model that generally does **not** live in Set:

Key Theorem

Given any geometric theory $\mathbb{T}$, there exists a **generic model** $G_{\mathbb{T}}$ from which all other $\mathbb{T}$-models can be obtained. It is generic in that it has no other properties other than the logical consequences of its being a model.

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Key Theorem

Given any geometric theory $\mathbb{T}$, there exists a **generic model** $G_{\mathbb{T}}$ from which all other $\mathbb{T}$-models can be obtained. In particular:

- $G_{\mathbb{T}}$ is *conservative*, i.e. for any geometric sequent σ

$$\forall \vec{x}.(\phi \longrightarrow \psi),$$

  σ holds for $G_{\mathbb{T}}$ iff σ holds for any $\mathbb{T}$-model.
- Any geometric construction (= built out of **arbitrary** colimits and **finite** limits) can be carried out for any specific $\mathbb{T}$-model.

Continuous Maps in Point-free Topology

Here's how we can use the generic model:

- Let $\mathbb{T}$ and $\mathbb{T}'$ be two geometric theories. Denote $[\mathbb{T}]$ and $[\mathbb{T}']$ to be their corresponding spaces of models.

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- Suppose we are able to transform $G_{\mathbb{T}}$ into a model of $\mathbb{T}'$ geometrically (= using finite limits & arbitrary colimits), which we denote $f(G_{\mathbb{T}})$. Notice: if $f(G_{\mathbb{T}})$ is a $\mathbb{T}'$ model, then $f(G_{\mathbb{T}}) \in [\mathbb{T}']$.

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- The same construction $f(-)$ extends to all other $\mathbb{T}$-models, and so what we have done is define a map

$$\begin{aligned} f \colon [\mathbb{T}] &\longrightarrow [\mathbb{T}'] \\ G_{\mathbb{T}} &\longmapsto f(G_{\mathbb{T}}) \end{aligned}$$

where $G_{\mathbb{T}}$ plays the role of the formal parameter.

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Example

Let $\mathbb{T}_{\mathbb{R}}$ be the theory of Dedekind reals, and denote $\mathbb{R}$ to be the space of reals, and $x \in \mathbb{R}$ to be the generic real. We can define:

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^2 \end{aligned}$$

Notice: No explicit continuity proof of this map is required – as long as we adhere to geometric constraints, we forgo the ability to define discontinuous maps. For a more detailed discussion, see [Vic22].

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Bird's Eye View

	Point-Set Topology	**Point-free Topology**
Point	Element of a set	Model of a theory $\mathbb{T}$
Space	Set + Topology	Universe of all $\mathbb{T}$-models
Continuous Maps	$f\colon X \to Y$ s.t. $f^{-1}(U)$ open, for all open U	Geom. transformation of generic model $G_{\mathbb{T}}$

Adelic Geometry via Topos Theory

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- **Observation #1**: Integer solutions imply real and modulo p solutions (in fact p-adic solutions).
- **Observation #2**: Real and p-adic solutions are easier to deal with than just integer/rational solutions.

- **New Question**: Given a polynomial with $\mathbb{Q}$-coefficients, when does knowledge about its $\mathbb{Q}_p$ and $\mathbb{R}$-solutions give us info about its $\mathbb{Q}$-solutions?

Local-Global Principle for $\mathbb{Q}$

Some property P is true for $\mathbb{Q}$ iff P is true for all the completions of $\mathbb{Q}$.

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Hasse-Minkowski Theorem: Quadratic forms[3] have $\mathbb{Q}$-solutions iff they have solutions over all completions of $\mathbb{Q}$.

Counter-Examples:

- Lind-Reichardt: $2Y^2 = X^4 - 17Z^4$
- Selmer: $3X^3 + 4Y^3 + 5Z^3 = 0$

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Still, it would be helpful to find a way of reasoning about properties that hold for all completions of $\mathbb{Q}$.

Classical Number Theory: Reason about completions via the adele ring

$$\mathbb{A}_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \left(\mathbb{R} \times \prod_{p} \mathbb{Z}_p \right)$$

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Point-Free Topology: Does there exist a geometric theory of completions of $\mathbb{Q}$? If yes, then there exists a generic completion which we can reason with.

Yet more foundational issues ...

When we say a property holds for all completions of $\mathbb{Q}$, we really mean true for all (non-trivial) completions of $\mathbb{Q}$ up to topological equivalence.

Places

A **place** is an equivalence class of absolute values whereby $|\cdot|_1 \sim |\cdot|_2$ if there exists some $\alpha \in (0,1]$ such that $|\cdot|_1 = |\cdot|_2^\alpha$ or $|\cdot|_2 = |\cdot|_1^\alpha$.

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In particular, if $|\cdot|_1, |\cdot|_2$ belong to the same place, then their respective completions of $\mathbb{Q}$ are equivalent.

Yet more foundational issues ...

Question: How should we think of the space of places of $\mathbb{Q}$?

Ostrowski's Theorem for $\mathbb{Q}$

Every absolute value of $\mathbb{Q}$ is equivalent to a (non-Archimedean) p-adic absolute value $|\cdot|_p$ (for some prime p), or the Archimedean absolute value $|\cdot|_\infty$.

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The Point-free Perspective

Let's rework this question via point-free topology.

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As long as we are careful to work geometrically, then our point-free spaces will maintain a tight connection with the more category-theoretic aspects from topos theory. This puts at our disposal a deep collection of structure theorems, such as descent, that allows us to extract topological information from our logical setup.

The Space of Places of $\mathbb{Q}$

Intuitively, what should this space look like?

Its points should correspond to equivalence classes of absolute values, such that:

1. $|\cdot|^\alpha \sim |\cdot|$

for any absolute value $|\cdot|$, and $\alpha \in (0,1]$.

Phrased more categeorically, we can define the space of places as the coequaliser of the diagram

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- $[av]$ is the space of absolute values of $\mathbb{Q}$
- π is the projection map sending $(|\cdot|, \alpha) \mapsto |\cdot|$
- ex is the exponentiation map sending $(|\cdot|, \alpha) \mapsto |\cdot|^\alpha$

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In essence, we would like to 'quotient' the topos [*av*] by an algebraic action – two questions:

- Is the notion of (real) exponentiation geometric? Ng-Vickers [NV22]
- What does it mean to quotient a (point-free) space by an algebraic action?

Non-Archimedean Place of $\mathbb{Q}$

$$\begin{tikzcd}[av_{NA}] \times (0, \infty) \arrow[r, bend left, "\pi"] \arrow[r, bend right, "ex"'] & {[av_{NA}]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D}\end{tikzcd}$$

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- For any non-Arch. absolute $|\cdot|$, exponentiating $|\cdot|^\alpha$ still yields a non-Arch. absolute value for any $\alpha \in (0, \infty)$.
- What is $\mathcal{D}$?

Theorem (N.-Vickers)

$\mathcal{D} \simeq \{*\}$

Archimedean Place

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- Can we play the same game as we did in the Non-Archimedean case? Answer: No! (The topos corresponding to $\mathcal{D}'$ has non-trivial forking in its sheaves)
- So what is $\mathcal{D}'$?

Theorem (N.-Vickers)

$\mathcal{D}' \simeq \overleftarrow{[0,1]}$, i.e. the space of 'upper reals' between 0 and 1.

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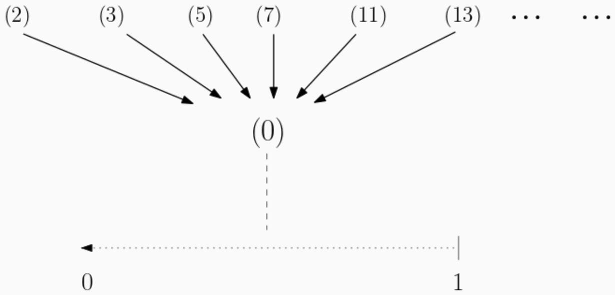


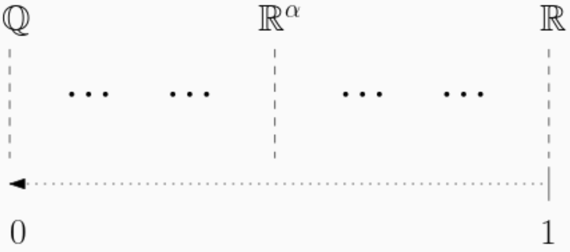
Discussion: Beauty & Danger of Analogies

"One weakness in the analogy between the collection of $\{K_s\}_{s\in S}$ for a compact Riemann surface S and the collection $\{\mathbb{Q}_p$, for prime numbers p, and $\mathbb{R}\}$ is that [...] no manner of squinting seems to be able to make $\mathbb{R}$ the least bit mistakeable for any of the p-adic fields, nor are the p-adic fields $\mathbb{Q}_p$ isomorphic for distinct p.

A major theme in the development of Number Theory has been to try to bring $\mathbb{R}$ somewhat more into line with the p-adic fields; *a major mystery is why $\mathbb{R}$ resists this attempt so strenuously."*

— Barry Mazur [Maz93]

Discussion: Connected & Disconnected



Reorienting our perspective

The issue of how to unite the Archimedean and the non-Archimedean settings is not (just) an algebraic question, but a topological one: how should the connected and the disconnected interact?

Proof of Theorem

Lax Descent Construction. Consider a 2-truncated simplicial topos $\mathcal{E}_\bullet$:

$$\begin{tikzcd} \mathcal{E}_2 \arrow[r, bend left, "\hat{d}_0"] \arrow[r, "\hat{d}_1"'] \arrow[r, bend right, "\hat{d}_2"'] & \mathcal{E}_1 \arrow[r, bend left, "d_0"] \arrow[r, bend right, "d_1"'] & \mathcal{E}_0 \arrow[l, "s_0"] \end{tikzcd}$$

We can obtain a category **LDesc**$(\mathcal{E}_\bullet)$ as the coinserter for the diagram (subject to the usual descent conditions). Its objects are pairs (F, θ), where:

- F is a sheaf of $\mathcal{E}_0$
- $\theta : d_0^* F \to d_1^* F$ is a morphism in $\mathcal{E}_1$ satisfying the unit and cocycle conditions.

Important: Unlike the standard descent topos, **no requirement** that θ is **isomorphism**!

Methodological challenge

The descent construction very much regards the topos as a category of objects, rather than a generalised space of models. To reformulate this in the point-free language, we decided to regard the sheaves as étale bundles, which keeps the connection with the point-free perspective.

Proof of Theorem

To prove the theorem, the basic plan of attack is to construct two functors

$$\begin{tikzcd} \mathfrak{J} : \mathcal{D}' \arrow[r, bend left] & \mathcal{S}[0,1] : \mathfrak{K} \arrow[l, bend left] \end{tikzcd}$$

where $\mathcal{D}'$ is the lax descent topos, and prove that they are inverse. The mathematical devil lies in the details.

- $\mathfrak{K}$ is induced by the fact that there exists a natural map from Dedekinds to upper reals defined by forgetting the left Dedekind section.
- $\mathfrak{J}$ is trickier, and involved constructing a technical lifting lemma, and showing that sheaves over $(0,1]$ restricted to the rationals $\mathbb{Q}_{(0,1]}$ (that obey the lax descent conditions) also satisfy the conditions of the lemma.

By way of conclusion

The main thread guiding us was a foundational question on the role of set theory in topology, and its broader effects on the foundations on other areas of mathematics.

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In different ways, we used the point-free perspective to pull these problems away from the underlying set theory. Both results indicate a particular loss of information within the classical setting, while also revealing a deep nerve connecting topology & algebra that had previously been obscured.

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